COMPUTATION OF REINSURANCE PREMIUMS BY INCORPORATING A COMPOSITE LORGNORMAL MODEL IN A RISK-ADJUSTED PREMIUM PRINCIPLE: APPLICATION TO GAM'S AUTOMOBILE INSURANCE BRANCH

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Abstract

This paper presents a formula for calculating a reinsurance premium which has been determined by incorporating a lognormal-burr model into a risk-adjusted premium calculating principle called the PH-transform principle. The lognormal-burr model has been selected and validated as the best fitting model for the used insurance data among the eight candidates of composite lognormal models. The formula has then been applied in calculating reinsurance premiums for an automobile insurance branch under an excess of loss non-proportional reinsurance treaty.

Keywords: risk-adjusted premium calculating principle, PH-transform premium principle, composite lognormal models, reinsurance premium formula, excess of loss non-proportional reinsurance treaty

1. Introduction

The risk faced by insurance companies in insuring goods and services is enormous. Claims may be made in such a way that the insurance company fails to pay them. You may take an example of an insurance company which may insure 10 houses from the same locality: the premium paid by the insured is usually less than the value of the property being insured, however, the insurance company promises to pay to the value of the property in case of damage. In the case where all the 10 houses are damaged completely due to, for example, an earthquake occurring in the locality, the insurance company is definitely not going to manage to pay all the claims which are going to be made. It is for cases as such that insurance companies go for reinsurance so that in case the claims are beyond paying them, the reinsurance company may come in to help.

The challenge that comes in now is how to calculate the reinsurance premium. Some reinsurance companies just multiply a certain rate to the total of collected premiums of a branch being reinsured and the result gotten is the reinsurance premium which they will require to be paid from the insurance company seeking reinsurance. However, this would not take into consideration the randomness nature of the claims being made. For this reason, actuaries have developed probabilistic models to be used in calculating reinsurance premiums. Premiums are calculated in such a way that data is first modelled to fit the candidate probability distributions. Then the best fitting probability distribution among the candidates is incorporated in an appropriate risk measure, requiring it, called the premium calculating principle.

Some of the risk measures called premium calculating principles that require fitting a probability distribution are the pure premium principle, expected value principle, variance principle, standard deviation principle, exponential principle, Esscher principle and the risk-adjusted premium principles [3].

Despite having been there so many desirable properties which a premium calculation principle must satisfy, Dickson (2005) lists most of the basic properties such as non-negative loading, additivity, scale invariance, consistence and no ripoff. And it was shown that the risk adjusted premium principle satisfies all except one, the property of additivity [3]. This definitely makes it one of the most desirable premium calculation principles to use for calculating either insurance or reinsurance premiums.

Now, the issue of which probability distribution to incorporate in the premium calculating principle is very serious. Distributions of insurance data have often been that of positive skew with a thick upper tail indicating a mixture of moderate and large claims [7]. In other words, they consist of what are called a head and a tail for moderate and large claims respectively [2]. Depending on whether moderate claims or large claims dominate the data, Calderin-Ojeda and Kwok (2015) explain that standard probability distributions such as Pareto, Gamma, Weibull, Lognormal and Inverse Gaussian have often been used to model the data. The only problem of modelling had often come in when the dominance of both moderate and large claims was apparent. To overcome this problem, several papers have proposed the use of composite probability models [7]. The composition of these models are two pieces of distributions separated at a certain threshold [16]. To model moderate claims for a head of a distribution, some papers have proposed the use of Weibull probability distribution [2] and others have proposed the use of lognormal [12]. Although Pareto has often been used to model the large claims for distributions' tails, some papers such as that of Nadarajar and Bakar (2012), have shown that Burr distribution modelled better when they worked with Danish Fire Insurance Data and also, the paper by Bakar, Hamzah, Maghsoudi and Nadarajar

(2015), have proposed the use of Burr, loglogistic, paralogistic, generalised pareto, pareto, inverse burr, inverse pareto and inverse paralogistic.

In the case where Weibull distribution is used in the head part, a composite model becomes a Composite Weilbull Model [2] and in the case where, instead of Weibull, we use lognormal, it becomes Composite lognormal model [12]; the composite model naming becomes dependent on the probability distribution modelling moderate claims, on the head.

Summarily, this paper presents section 2 with the type of risk-adjusted premium principle (the PH-transform principle) which will be used for reinsurance premium calculation, the general structure in terms of probability density function (5) and cumulative distribution function (9) for the two-piece composite probability distribution models which will be able to take lognormal for the head and seven diverse other probability distributions for the tail in section 3, some characteristics of the excess of loss reinsurance treaty essential for this study, the two reinsurance premium formulae and the methods used in selecting and validating the best among the candidate composite models. Section 3 presents the fitting of the candidate models to the data, selecting and validating the best fitting models, incorporation of the best and valid composite model into the reinsurance premium formulae (12) and (13), and the computation of the reinsurance premiums at diverse values of treaty priorities (retentions) while adjusting the risk aversion index thrice. Section 4 presents the evaluation of results obtained in section 3, section 5 presents some limitations to some of the mathematical methods used in this paper and section 6 presents all the general tools and techniques used in this paper to arrive at our results.

2. Theoretical Framework

2.1 *Risk-Adjusted Premium Calculation Principle*

Given that X is a non-negative random variable representing insurance claims, its survival function will be given by $S_X(x) = P(X > x) = 1 - F(x)$ [3], [9].

By using transforms to distort X's survival function as $S_Z(x) = g(S_X(x))$, Wang(1996b) proposed a general class of premium calculation principles given by

$$\Pi_X = \int_0^{+\infty} g(S_X(x))dx \qquad (1)$$

where the function g:

- is increasing, continuous, concave, and we have $g(0) = 0$ and $g(1) = 1$ [9].

The premium calculation principles we get from the above general class are called the *risk adjusted premium principles*. When the survival function is distorted by having $g(x) = x^{1/r}$, we get a risk adjusted premium principle called the *proportional hazard transform* (PH-transform) *principle* given by

$$\Pi_r(X) = \int_0^{+\infty} \left(S_X(x)\right)^{1/r} dx \qquad (2)$$

where $r \geq 1$, and r is referred to as a *risk aversion index* [9].

2.2 *Two-Piece Composite Models Structure in Terms of PDF and CDF*

Knowing that the survival function $S_X(x)$, in equation (2), equals $1 - F(x)$, it is evident that the challenge is in determining the better fitting $F(x)$ and this paper proposes the use of composite lognormal Models which were proposed by Nadarajar and Bakar (2012).

Nadarajar and Bakar (2012) present a general two probability density function of composite models in the form

$$f(x) = \begin{cases} f_{Comp1}(x) & , \quad if \ -\infty < x \leq \theta \\ f_{Comp2}(x) & , \quad if \ \theta < x < +\infty \end{cases} \qquad (3)$$

getting equated to

$$f(x) = \begin{cases} a_1 f_1^*(x) & , \quad if \ -\infty < x \leq \theta \\ a_2 f_2^*(x) & , \quad if \ \theta < x < +\infty \end{cases} \qquad (4)$$

Where $f_{Comp1}(x)$ is standing for the head part of the distribution modelling moderate claims while being taken as a lognormal probability density function and $f_{Comp2}(x)$ is standing for the tail part of the distribution modelling large claims while it can be taken by diverse probability density functions such as pareto, inverse pareto, burr, inverse burr, paralogistic, inverse paralogistic and loglogistic,

From equation (4), we have θ representing a threshold at which the distribution modelling moderate claims separates from a distribution modelling large claims. Also, we have $f_1^*(x) = \frac{f_1(x)}{F_1(\theta)}$ and $f_2^*(x) = \frac{f_2(x)}{\{1-F_2(\theta)\}}$. The non-negative weights a_1 and a_2 are factors of normalisation given by $a_1 = \frac{1}{1+\phi}$ and $a_2 = \frac{\phi}{1+\phi}$ such that, for $\phi = \frac{f_1(\theta)[1-F_2(\theta)]}{f_2(\theta)F_1(\theta)} > 0$ we have $a_1 + a_2 = 1$.

The probability density function (4) must be continuous and differentiable at the threshold θ and to be sure that these properties are always satisfied, Nadarajar and Bakar (2013) imposed

$a_1 f_1^*(\theta) = a_2 f_2^*(\theta)$ and $a_1 \frac{df_1^*(\theta)}{d\theta} = a_2 \frac{df_2^*(\theta)}{d\theta}$.

In replacing the above expressions for $f_1^*(x)$, $f_2^*(x)$, ϕ, a_1 and a_2 into equation (4), we obtain [16]

$$f(x) = \begin{cases} \frac{1}{1+\phi} \frac{f_1(x)}{F_1(\theta)}, & si \ -\infty < x \leq \theta \\ \frac{\phi}{1+\phi} \frac{f_2(x)}{\{1-F_2(\theta)\}}, & si \ \theta < x < +\infty \end{cases} \qquad (5)$$

The cumulative distribution function (cdf) $F(x)$ will be obtained by integrating the pdf (5) in order to have an expression of the form

$$F(x) = \begin{cases} F_{Comp1}(x) & , \quad si \ -\infty < x \leq \theta \\ F_{Comp2}(x) & , \quad si \ \theta < x < +\infty \end{cases} \qquad (6)$$

such that $F(x)$ satisfies the following properties [11]:

- $0 \leq F(x) \leq 1$;
- F is non-decreasing, that is to say if $x < y$, then F$(x) < F(y)$;
- $\lim_{x\to+\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$;
- F is continuous to the right, that is to say
  $\lim_{y\downarrow x} F(y) = F(x)$

By integrating each piece of (5) separately, we have

$$F_{Comp1}(x) = \int_{-\infty}^{x} f_{Comp1}(t)\,dt = \frac{1}{1+\phi}\frac{1}{F_1(\theta)}\int_{-\infty}^{x} f_1(t)\,dt = \frac{1}{1+\phi}\frac{1}{F_1(\theta)}\left[F_1(x) - \lim_{t\to-\infty} F_1(t)\right] = \frac{1}{1+\phi}\frac{1}{F_1(\theta)}[F_1(x) - 0] = \frac{1}{1+\phi}\frac{F_1(x)}{F_1(\theta)} \quad (7)$$

and

$$F_{Comp2}(x) = F_{Comp1}(\theta) + \int_{\theta}^{x} f_{Comp2}(t)\,dt = \frac{1}{1+\phi}\frac{F_1(\theta)}{F_1(\theta)} + \frac{\phi}{1+\phi}\frac{F_2(x)-F_2(\theta)}{\{1-F_2(\theta)\}} = \frac{1}{1+\phi} + \frac{\phi}{1+\phi}\frac{F_2(x)-F_2(\theta)}{\{1-F_2(\theta)\}} = \frac{1}{1+\phi}[1 + \phi\frac{F_2(x)-F_2(\theta)}{\{1-F_2(\theta)\}}] \quad (8)$$

Thereby giving the following cumulative distribution function

$$F(x) = \begin{cases} \frac{1}{1+\phi}\frac{F_1(x)}{F_1(\theta)} \ , & si\ -\infty < x \leq \theta \\ \frac{1}{1+\phi}\left[1 + \phi\frac{F_2(x)-F_2(\theta)}{1-F_2(\theta)}\right] \ , & si\ \theta < x < +\infty \end{cases} \quad (9)$$

2.3 *Excess of Loss Non-Proportional Reinsurance Treaty*

This treaty has characteristics such as treaty priority, treaty guarantee and treaty ceiling [6].

2.3.1 *The Treaty Priority*

The priority also called the retention R is the agreement's claim amount at which a Reinsurer intervenes provided the claim or claims of an event amount equals or exceeds R [10].

2.3.2 *The Treaty Guarantee*

The treaty guarantee or the limit h is the agreed amount exceeding R beyond which the Reinsurer does not intervene in paying the claim or claims of an event. This means the Reinsurer is obliged to pay a claim or claims of an event exceeding R but this claim (or claims of event) must be less than or equal to h [6]. However, it must be noted that some excess of loss non proportional reinsurance treaties have a treaty guarantee without limit, i.e. $h = +\infty$ [13].

2.3.3 *The Treaty Ceiling*

The treaty ceiling $R + h$ is an amount of a claim or claims of an event beyond which the Reinsurer does not intervene [6]. This means the Reinsured is itself responsible to pay an amount of claim in excess of $R + h$. Also to be noted that in case of limitless treaty guarantee, the Reinsurer is responsible to pay any amount exceeding the retention R.

2.3.4 *Reinsurer's Responsibility in Limitless Treaty Guarantee Case*

In this case, where X is a random variable for a claim or claims of event amount, the amount $L_{h\to+\infty}$ to be paid by the Reinsurer is presented as

$$L_{h\to+\infty}(X) = \begin{cases} 0, & if \quad 0 \le X < R \\ X - R & if \quad R \le X \end{cases} \tag{10}$$

2.3.5 *Reinsurer's Responsibility in Limited Treaty Guarantee Case*

In this case, X being a random variable for claims, the amount to be paid or contributed towards payment of a claim or claims of an event amount is given by [9]

$$L_h(X) = \begin{cases} 0, & if \quad 0 \le X < R \\ X - R & if \quad R \le X < R + h \\ h & if \quad X \ge R + h \end{cases} \tag{11}$$

2.4 *Reinsurance Premium*

The premium calculation principle (2) is a perfect example for calculating a reinsurance premium if it were assumed that any amount made as a claim was to be paid by the Reinsurer and the excess of loss non proportional reinsurance treaty does not contain any such characteristics as treaty priority, treaty guarantee and treaty ceiling: this can be observed by the integration being carried out between 0 and $+\infty$. However, in the presence of treaty priority, treaty guarantee and treaty ceiling, the reinsurance premium will be calculated as follows [9]:

2.4.1 *Reinsurance Premium in Limitless Treaty Guarantee Case*

$$\Pi_r(R) = \int_R^{+\infty} \left(S_X(x)\right)^{1/r} dx \qquad (12)$$

2.4.2 *Reinsurance Premium in Limited Treaty Guarantee Case*

$$\Pi_r(R) = \int_R^{R+h} \left(S_X(x)\right)^{1/r} dx \qquad (13)$$

2.4.3 *Reinsurance Premium where F(x) is Composite Lognormal Distribution*

This will be determined after having selected a best fitting composite lognormal distribution model to data in section 3. Before presenting section 3, we present tools for selecting a best fitting probability distribution to data in section 2.5.

2.5 *Tools for Selecting a Best Fitting Probability Distribution Model*

2.5.2 *Estimation of Parameters and Classification of Candidate Models*

Maximum Likelihood Estimator (MLE)

By maximum likelihood method, we estimate parameters of a given probability distribution by differentiating the function $logL(\theta; x_1, x_2, x_3, \dots, x_n)$, called the log-likelihood, in terms of parameters being represented by θ. The derivatives are then equated to zero and then we solve for the parameters [5]. Usually, this method takes long or is complicated to use, as a result, we apply to it numerical methods to arrive at estimated parameters, see [2], [12]. Please take note that

- $(x_1, x_2, x_3, \dots, x_n)$ is a sample of n observed random variables which are independent and of the same probability distribution.
- $L(\theta; x_1, x_2, x_3, \dots, x_n)$ is a joint probability distribution function called joint cumulative distribution function if $(x_1, x_2, x_3, \dots, x_n)$ is discrete or joint probability density function if $(x_1, x_2, x_3, \dots, x_n)$ is continuous. It is called a likelihood function and is considered as a function of only θ.
- θ is a set of all parameters of a given probability distribution.

After estimations, estimated parameters come along with a value called the log-likelihood value.

Akaike Information Criterion (AIC)

The criterion is used to measure the quality of a model by penalising the model in terms of its number of parameters. It is most suitable to use only for classification purposes of distributions than for making decisions [13]. And the model with the smallest AIC is classified as the best. The AIC is given by

$$AIC = 2NLL + 2k$$

Where k is a number of parameters to estimate for the model and NLL is a negative log-likelihood value.

2.5.1 *Goodness of fit Tests*

We will use the goodness-of-fit tests as used by Calderin-Ojeda and Kwok (2015). They defined goodness-of-fit measures as test statistics that quantify the 'distance' between empirical distribution function (EDF) constructed from the data and the cumulative distribution function (cdf) of the fitted models. Based on the work of Rizzo (2009), they suggested the use of

- Kolmogorov-Smirnov (KS) test statistics given by $D = \max(D^+, D^-)$, where $D^+ = \max\limits_{1\le j\le N}\left\{\frac{j}{N} - \hat{F}(x_{(j)})\right\}$ and $D^- = \max\limits_{1\le j\le N}\left\{\hat{F}(x_{(j)}) - \frac{j-1}{N}\right\}$

- Cramer-von Mises (CvM) test statistic given by $W^2 = \sum_{j=1}^{N}\left[\hat{F}(x_{(j)}) - \frac{2j-1}{2N}\right]^2 + \frac{1}{12N}$

- Anderson-Darling (AD) test statistic given by $A^2 = -N - \frac{1}{N}\sum_{j=1}^{N}[(2j-1)\log\left(\hat{F}(x_{(j)})\right) + (2n+1-2j)\log(1-\hat{F}(x_{(j)}))]$

Where,

$\hat{F}$ is the cdf of the fitted model, $x_1, x_2, \dots, x_N$ is the original data and $x_{(1)}, x_{(2)}, \dots, x_{(N)}$ is an increasing ordered data from the original data. And the smaller the values of KS, CvM and AD are, the better the model fits the data [7].

2.5.2 *Model Validating*

To prove the validity of the model selected by the indication of the goodness-of-fit tests as best, we will carry out a hypothesis test with the following hypothesis

Null hypothesis (H_0): the best model is valid if its p-value is greater than the level of significance α.

Alternative hypothesis (H_a): the best model is not valid if its p-value is less than the level of significance α.

In this paper, we will use the standard level of significance $\alpha = 0.05$.

We will again proceed with Calderin-Ojeda and Kwok (2015) approach of determining the p-value using the bootstrap procedure in the following order:

- For the model selected as a better fit to the data using methods in section 2.5.1, calculate the goodness-of-fit test statistics t_{KS}, t_{CvM} and t_{AD},
- Using the model providing a better fit to data $x_1, x_2, \dots, x_N$,
  - generate M sets of resampled data and denote them as $\hat{x}_1^{(i)}, \hat{x}_2^{(i)}, \dots, \hat{x}_N^{(i)}$ for $i = 1, \dots, M$.
  - refit it to each set of the resampled data and then compute the test statistics $t_{KS}^{(i)}, t_{CvM}^{(i)}$ and $t_{AD}^{(i)}$ for $i = 1, \dots, M$.
- Then, finally, determine the p-values by

$\frac{\#\left\{i: t_{KS}^{(i)} \geq t_{KS}\right\}}{M}$, $\frac{\#\left\{i: t_{AD}^{(i)} \geq t_{AD}\right\}}{M}$ and $\frac{\#\left\{i: t_{CvM}^{(i)} \geq t_{CvM}\right\}}{M}$

In section 3, the resampling will be done by taking $M = 1000$.

3. Theoretical Applications to Insurance Claims Data

The data for the applications is that of all automobile insurance claims made in 2016 for GAM insurance company. Claims were of two types: corporal claims and materials claims. Corporal claims are claims made on damages caused directly to persons' bodies and material claims are claims made on damages caused to vehicles. The data had a total of 6499 claims made, from which 0.4% were corporal claims and 99.6% were material claims.

The company had entered into a 2017 excess of loss non proportional reinsurance treaty with some of the characteristics being as follow:

- Treaty priority = 10 000 000 DZD per claim amount (or event's total claim amount)
- Treaty ceiling:
  - unlimited for corporal damages
  - limited to 150 000 000 DZD for material damages.

3.1 *Composite Lognormal Model Fitting to Data*

We made available seven candidate composite lognormal models for fitting to data: Lognormal-Pareto, lognormal-burr, lognormal-inverse paralogistic, lognormal-paralogistic, lognormal-loglogistic, lognormal-inverse pareto and lognormal-inverse burr. When, for example, the tail part of the distribution (modelling large claims) was to be modelled by Pareto distribution, the composite lognormal model became composite lognormal-Pareto (LPC) probability distribution model.

The fitting, basically, produced the estimated parameters (whose exponentials are the values put in table 3.1) and the negative log-likelihood (NLL) of which (–NLL) was used to calculate the Akaike Information Criterion (AIC) [2]. Latter, we calculated the KS, CvM and AD test statistics to aid in the selection of a model providing a better fit as we could not entirely rely on AIC.

				Goodness-of-fit Test Statistics		
Model	**Parameters**	**Estimated Parameters**	**AIC**	**KS**	**CvM**	**AD**
Lognormal-Pareto (LPC)	β θ	0.7181631 10233.05	143141.2	0.1288046	22266.22	198.393
Lognormal-Burr (LBC)	σ θ α β s	1.117488 17714.93 0.03118409 51.30768 16627.52	141001.1	0.06464014	1327.889	22.9029
Lognormal-Inverse Paralogistic (LIPaC)	σ θ τ s	0.8050095 37399.35 1.610152 1.754455	141236.2	0.2708801	32411.05	360.5476
Lognormal-Paralogistic (LPaC)	σ θ α s	0.80475 37335.19 1.267696 0.2210377	141236.2	0.06385445	379.0071	27.40241
Lognormal-loglogistic (LLC)	σ θ γ s	0.8056159 37384.58 1.605893 5.182061	141236.2	0.06411453	396.6818	27.56333
Lognormal-Inverse Pareto (LIPC4)	σ θ α s	0.7786674 21911.52 0.2620021 0.0947796	141647.4	0.08487503	10560.21	86.81077
Lognormal–Inverse Burr (LIBC)	σ θ τ γ s	0.8059115 37448.29 13.68639 1.607251 0.2197676	141238.2	0.06419904	410.3249	27.59355

Table 3.1: Fitting Composite Lognormal Probability Distribution Models to Insurance Claims

We take note that the lognormal-burr(LBC) model produces the smallest value of AIC, hence, we conclude that it has been classified first. And the AD test statistic supports that it should be considered as a model providing a better fit to the insurance claims data despite the KS and CvM test statistics having gone for lognormal-paralogistic ($LPaC$).

3.2 *Testing the Validity of LBC and $LPaC$ Models*

The validity of each model will be tested based on the hypotheses

$$H_0: the\ model\ is\ valid\ if\ p-value\ \geq\ \alpha = 0.05$$

$$H_a: the\ model\ is\ not\ valid\ if\ p-value\ <\ \alpha = 0.05$$

Where $\alpha = 0.05$ is a level of confidence which means that of all the calculations we will make, we have a chance of $1-\alpha = 0.95$ that they are going to be correct with a chance of $\alpha = 0.05$ that they are incorrect if the hypothesis H_0 is accepted.

As shown in section 2.5.2, the p-values will be determined in terms of KS, CvM and AD test statistics using the bootstrap method where $M = 1000$.

	p-values		
Model	**KS**	**CvM**	**AD**
Lognormal-Paralogistic (LPaC)	0.702	0.544	0.593
Lognormal-Burr (LBC)	0.643	0.497	0.628

Table 3.2: p-values of best fitting Composite Lognormal Models

We take note that all the p-values being above 0.05 signifies that all the two models have been accepted as being valid.

3.3 *Incorporating LBC into the PH-transform Principle*

Although both the lognormal-burr and the lognormal-paralogistic qualify as models giving a better fit to data, we opt to use the lognormal-burr in the reinsurance premium principle (12) and (13) because it would take much more space if we used all the two. Also, some other reasons are given in section 4 for the preference of lognormal-burr to lognormal-paralogistic.

Using cumulative distribution function (9), we take $F_1(x) = \Phi\left(\frac{lnx-\mu}{\sigma}\right)$, where $\mu, \sigma > 0$, as a cumulative distribution function for lognormal and we take $F_2(x) = 1 - (1 + (x/_s)^{\beta})^{-\alpha}$, where $\alpha, \beta, s > 0$, as a cumulative distribution function for lognormal for burr. Consequently, we have a cumulative distribution function for composite lognormal-burr probability distribution model given by

$$F(x) = \begin{cases} \frac{1}{(1+\phi)} \frac{\Phi((lnx-\mu)/\sigma)}{\Phi((ln\theta-\mu)/\sigma)}, & si\ 0 < x \le \theta \\ 1 - \frac{\phi}{1+\phi}\left(\frac{1+\left(\frac{\theta}{s}\right)^{\beta}}{1+\left(\frac{x}{s}\right)^{\beta}}\right)^{\alpha}, & si\ \theta < x < +\infty \end{cases} \quad (14)$$

The reinsurance premium formula, using the PH-transform principle, is determined by,

$$\Pi_r(R) = \int_R^{+\infty} (S(x))^{\frac{1}{r}} dx = \left(\frac{\phi}{1+\phi}\right)^{\left(\frac{1}{r}\right)} * \left(1 + \left(\frac{\theta}{s}\right)^{\beta}\right)^{(\alpha/r)} * \int_R^{+\infty} \frac{1}{\left(1+\left(\frac{x}{s}\right)^{\beta}\right)^{\alpha/r}} dx, \text{ for } \theta < R, \quad (15)$$

provided, of course, that $S(x) = 1 - F(x) = \frac{\phi}{1+\phi}\left(\frac{1+\left(\frac{\theta}{s}\right)^{\beta}}{1+\left(\frac{x}{s}\right)^{\beta}}\right)^{\alpha}$, $where\ \theta < x < +\infty$.

The integral in formula (15) exist if and only if $1 \le r < \alpha * \beta$. Nadarajar and Bakar (2012) show that the parameters μ and φ can be calculated from the other estimated parameters by $\mu = ln\theta + \sigma^2 + \theta\sigma^2 \frac{f_2'(\theta)}{f_2(\theta)}$ and $\phi = \frac{(\theta^{\beta}+s^{\beta})\psi[(ln\theta-\mu)/\sigma]}{\sigma\alpha\beta\theta^{\beta}\Phi[(ln\theta-\mu)/\sigma]}$.

Formula (15) is suitable for limitless guarantee case and in the case where the guarantee is limited, the reinsurance premium will be given by

$$\Pi_r(R) = \int_R^{R+h} (S(x))^{\frac{1}{r}} dx = \left(\frac{\phi}{1+\phi}\right)^{\left(\frac{1}{r}\right)} * \left(1 + \left(\frac{\theta}{s}\right)^{\beta}\right)^{(\alpha/r)} * \int_R^{R+h} \frac{1}{\left(1+\left(\frac{x}{s}\right)^{\beta}\right)^{\alpha/r}} dx,$$

$$for\ r \ge 1\ and\ \theta < R, \quad (16)$$

where, of course, R is a treaty priority and h is a treaty guarantee [9].

Due to corporal claims under unlimited guarantee being 0.4% and material claims under limited guarantee being 99.6% of the whole claims, we decided to consider corporal claims conditions for reinsurance cover as negligible in the presence of those for material claims. As a result, formula (16), instead of formula (15), was used for premium calculations. The reinsurance premium computations using formula (16) were done numerically in R statistical software [15], hence, one of the reasons as to why formula (16) was left without completing the integration.

3.4 *Reinsurance Premium Computations*

They have been computed at diverse values of retention R and risk aversion index r.

	Reinsurance Premiums		
	Risk Aversion Index $\boldsymbol{r = 6.8}$	**Risk Aversion Index** $\boldsymbol{r = 7}$	**Risk Aversion Index** $\boldsymbol{r = 10.8}$
Retention (Treaty Priority) R	**Reinsurance Premium** $(\boldsymbol{\Pi_{r=6.8}(R)})$	**Reinsurance Premium** $(\boldsymbol{\Pi_{r=7}(R)})$	**Reinsurance Premium** $(\boldsymbol{\Pi_{r=10.8}(R)})$
1 000 000	19 869 439	21 030 169	41 637 318
2 000 000	19 561 745	20 711 946	41 161 331
4 244 000	18 980 106	20 107 452	40 202 598
5 000 000	18 802 541	19 922 384	39 898 947
6 490 000	18 469 874	19 575 158	39 319 291
8 000 000	18 150 743	19 241 536	38 751 796
10 000 000	17 749 056	18 820 999	38 023 887
12 000 000	17 365 970	18 419 392	37 317 382
14 000 000	16 997 693	18 032 873	36 628 193
16 000 000	16 641 634	17 658 818	35 953 489
18 000 000	16 295 925	17 295 329	35 291 199
20 000 000	15 959 157	16 940 976	34 639 746

Table 3.3: Reinsurance Premiums at diverse values of retention (R) and risk aversion index (r)

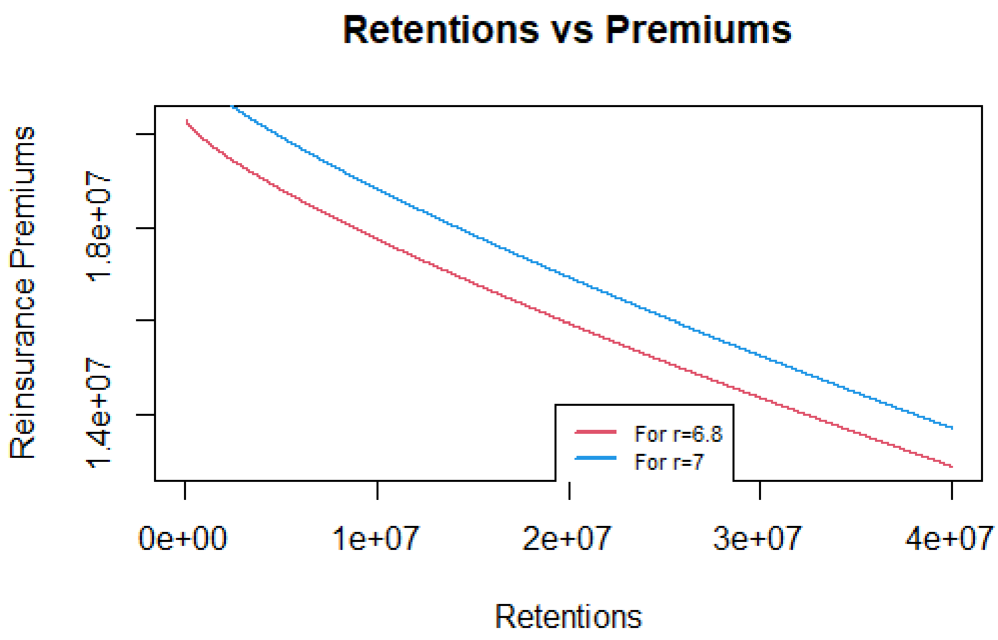

Figure 3.4: Plots of Retentions and Reinsurance Premiums for $r = 6.8$ and $r = 7$

4. Evaluation

Among the risk adjusted premium principles of Wang(1996b), we have chosen to use the PH-transform principle because it provides for the treaty priority and treaty ceiling in the calculation of the reinsurance premium under the excess of loss non proportional reinsurance treaty [9], as can be seen in **table 3.3**. The PH-transform principle also provides for the possibility of adjusting the risk aversion index depending on whether the reinsurer anticipates the high or low risk on damage claims because the premium increases as the risk aversion index increases and vice versa as is shown by **figure 3.4** and **table 3.3**. As is also shown in **figure 3.4**, the higher the treaty priority the lower the calculated PH-transform reinsurance premium [9]. **Table 3.1** shows that the composite lognormal-burr model is best of the candidate models due to the smallest values of AIC and AD test statistic. And going by the KS and CvM test statistics, the composite lognormal-paralogistic model is being presented as the one providing the best fit to data. However, when we compare the KS test statistics for lognormal-burr and lognormal-paralogistic we see that there is very minimal difference which suggests that the KS could have favoured the lognormal-burr except that it is sensitive in capturing the behaviour of the model at the tail [7]. Having the possibility of lognormal-paralogistic not being supported as the best of the candidate models does not make it an invalid model or lesser best fitting model as is evidenced by the p-values in **table 3.2**. They both can be just as best fitting models except that, also, going by the suggestions of the values of AIC and AD test statistic, and the high chance of the KS being not so reliable as to capture the models behaviour at the tail, we opted to use the lognormal-burr model in our reinsurance premium formulae (15) and (16). Also, the possibility of some integrals not being able to exist for some composite models led us to leave the reinsurance formulae (15) and (16) in un integrated form in order for it to be solved by computing numerically.

5. Limitations

The PH-transform principle though very desirable in the property of premiums adjustments because of the presence of the risk aversion index cannot be used in modelling and computations of reinsurance premiums for all reinsurance treaties. Reinsurance treaties such as the surplus proportional reinsurance treaty [10] and those that do not involve treaty priority would require a research of other premium calculation principles suitable for them. Also, despite the composite lognormal models having produced best fitting models from among them, there still stands a chance that other composite models, such those that would use Weibull instead of lognormal [2], would still produce a much better fitting model. Therefore, the composite lognormal-burr model having come out as best in this paper does not imply it is the best of any models that may be fitted to the given data. Only space in this paper and time has limited us from presenting other possible candidate models.

6. General Tools and Techniques

For all the computations in this paper, we used the R statistical software version 4.1.0 [15]: The parameters were estimated by the optim function in collaboration with the dcomplnorm function of the CompLognormal package [12]. The negative log-likelihood (NLL) was part of the results gotten from the parameter estimation process and we manually calculated the AIC by incorporating the –NLL in the formula for AIC [2]. The seven probability distributions used to model the tail are of the family of transformed beta distributions that we got from the actuar package [2], [4]. The test statistics KS, CvM and AD were computed by programming their corresponding functions suitable for their computations. The p-values related to KS, CvM and AD test statistics were also computed by programming related functions for their computations using other functions such as sample and the already created functions for KS, CvM and AD test statistics. The numerical integration of the reinsurance premium formula (16) was done by using the integrate function. And the retention vs premiums figure was created using plot, lines and legend functions after programming a function which gives a number of reinsurance premiums given a number of retentions.

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