The Negation of Impossibility

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Abstract

The impossibility theorems of Arrow and Gibbard-Satterthwaite have been thought to rule out economic democracy and welfare economics. This paper demonstrates an information processing system which accords with the premises of these authors, and, consequently, proves their conclusions of impossibility to be untrue except as mathematical tautologies.

The Negation of Impossibility

Abstract

In 1951 Kenneth Arrow published a book in which he proved that social choice was impossible. There was no way to amalgamate individual preference orderings into a social preference ordering in such a way that certain rational and normative conditions were met. Later Gibbard and Satterthwaite proved that any such amalgamation of individual preference orderings in which there was no advantage to any individual to use strategy to order their preferences insincerely in order to get a better result for themselves was impossible or led to the selection of a dictator. These impossibility theorems have been thought to rule out economic direct democracy and also welfare economics giving credibility to the implication that representative democracy and capitalist economics are the best systems that can be devised.

Instead of simple amalgamation, we have devised a more general information processing system which accepts inputs from individual choosers as either preference orderings or ratings and outputs a social choice which may consist of one or more outcomes and can be in the form of either cardinal or ordinal information. This system is utility based but processes the information in such a way as to alleviate concerns about interpersonal comparisons of utility. It is a hybrid utilitarian approval choosing system. Instead of the individual choosers using strategy, the system itself maximizes the efficacy of each individual input thus disincentivising individuals from choosing insincerely. It also meets Arrow's five rational and normative conditions thus proving that social choice is not impossible. The result is that a utility based social choice system has been devised which negates both impossibility theorems and should give new life to welfare economics and economic direct democracy.

Introduction

In Social Choice and Individual Values, Kenneth Arrow wrote[1], “In a capitalist democracy there are essentially two methods by which social choices can be made: voting, typically used to make ‘political’ decisions, and the market mechanism, typically used to make ‘economic’ decisions.” He goes on to say, “The methods of voting and the market … are methods of amalgamating the tastes of many individuals in the making of social choices.” Initially, Arrow does not distinguish between political and economic systems claiming that both are means of formulating social decisions based on individual inputs. Arrow then purports to show that there is no rational way to make social decisions based on the amalgamation of individual ones thus ruling out welfare economics or economic democracy and also direct political democracy. The dichotomy between political and economic systems remains with the implication being that representative democracy and capitalist economics are the best systems that can be devised.

Gibbard[2] and Satterthwaite[3] concurred with Arrow and proved that any social choice system that was strategy proof was also impossible. Gibbard stated: “An individual manipulates a system of voting if, by misrepresenting his preferences, he secures a result he prefers to the result that would obtain if he expressed his true preferences.” Satterthwaite showed that the requirement for choosing procedures (what he called voting procedures) of strategyproofness and Arrow’s requirements for social welfare functions are equivalent: a one-to-one correspondence exists between every strategy-proof voting procedure and every social welfare function satisfying Arrow’s five requirements.

A major stumbling block for the development of rational social choice systems regards the issue of interpersonal comparisons. It has been thought that scales which measure the utilities of individuals are incompatible, and that any scale chosen upon which all individuals were supposed to rate their utilities would be arbitrary. Arrow states[4]: “If we admit meaning to interpersonal comparisons of utility, then presumably we could order social states according to the sum of utilities of individuals under each, and this is the solution of Jeremy Bentham, accepted by Edgeworth and Marshall.” He also states[5]: “The viewpoint will be taken here that interpersonal comparison of utilities has no meaning and, in fact, that there is no meaning relevant to welfare comparisons in the measurability of individual utility.” Thus, according to Arrow, any individual input must be based on individual preference rankings of the form aPbPc ... meaning a is preferred to b is preferred to c etc. or the notation Arrow uses – aRbR, a is preferred or equal to b etc. –...and not utility ratings.

Utilities can be measured on a scale such as the real line from 1 to 100 for example. This could be symbolized as the set $\{U| u_1 \leq u_i \leq u_{100}\}$. In general there will be a utility u_i for each possible alternative. A set of candidates or alternatives (whether of the political or economic variety) can be associated with a set of utilities, and, if $u_a > u_b$, meaning the utility of alternative a is greater than the utility of alternative b, then aPb. A set of utilities will produce preference ratings while a relation of the form aPb is considered a preference ranking.

We develop a social choice system that is utility based, but which overcomes the objections of arbitrariness of utility scales, is strategyproof and also meets Arrow's five normative and rational criteria.

Utilitarian and Approval Choosing

Approval choosing is exactly analogous to approval voting and, therefore, “voting” and “choosing” are used interchangably for the purposes of this paper.

It has been shown by Aki Lehtinen[6] that strategic voting behavior which violates one of Arrow's conditions is actually beneficial, and, therefore, one of Arrow's conditions is not normatively acceptable. Lehtinen asserts: “This means that, while Arrow’s theorem and the Gibbard-Satterthwaite [theorem] are logically impeccable, they fail to have the devastating consequences for democracy that have sometimes been attributed to them.”

Arrow sets up the problem so that each individual voter or chooser orders all alternatives and then society is required to come up with an ordering that is best according to some criteria. He states[7] “In the theory of consumer's choice each alternative would be a commodity bundle; ... in welfare economics, each alternative would be a distribution of commodities and labor requirements. … in the theory of elections, the alternatives are candidates.” Insofar as voting is concerned, there will be winners and losers. As Smith[8] has pointed out, we don't need to be concerned about the ordering of the losers. “Nobody cares about rank-ordering the losers! We care about finding the winner.” Or winners in the case of multi-winner elections. However, the method constructed in this paper actually inputs information from the individual choosers which can be either in the form of rankings or ratings and outputs information in the form of both complete social rankings and/or ratings including ordering of the losers!

As inputs Arrow insists on orderings instead of the more nuanced cardinal utility based information in order to avoid interpersonal comparisons. However, voting, *ergo facto*, is a process in which all are assumed interpersonally comparable in terms of one person, one vote. The same rationale is assumed for economic decision making by many writers. Lehtinen[9] asserts: “If the principle is justified by appealing to interpersonal comparisons, the weight of each individual in determining the social optimum ought to be the same. The justification of the one-man-one-vote principle derives here from the claim that, a priori, each individual ought to have equal weight in determining the will of the people. It follows that each individual ought to have the same opportunity to affect the outcome of a voting process. Another way to look at the issue is to note that since the one-man-one-vote principle is violated only if we know that some voters have a legitimate claim to more than equal influence on the voting outcome, when there are no such reasons to violate the principle, we should also assume that each voter’s utility is measured with the same scale.”

In fact Arrow's assumption of input preference orderings or rankings for each individual is a tacit assumption of equal utility scales for each individual. With the assumption that orderings represent equally spaced utilities, we can convert orderings or rankings to ratings and vice versa at least for the information processing system considered here. Without loss of generality any scale can be used for this procedure as long as the end points represent the most preferred and least preferred alternatives..

The method we describe involves placing a threshold in an optimal manner for each chooser such that all alternatives with corresponding sincere utilities above that threshold are given positive approval style choices. Alternatives with corresponding utilities below threshold are given negative approval style choices. This manner of approval "voting" is considered sincere with respect to Niemi's[10] definition of sincere approval voting. As Niemi points out, "... under AV sincere voters are still left with multiple strategies to consider." Not all strategy is considered insincere i.e. approval voting is not strategyproof according to him. However, since there is an optimal strategy for each individual, if the information processing system itself applies that strategy instead of the individual choosers doing it, there is no incentive for an individual to use strategy. In fact it would only diminish the outcome for them personally.

Regarding Arrow's condition, Independence of Irrelevant Alternatives (IIA), Lehtinen[11] has shown that IIA is moot if strategy is involved. "However, from the utilitarian and thus welfarist point of view, strategic voting is desirable rather than undesirable under most commonly used voting rules." Cox[12] has also considered strategic voting in multi-winner districts. Even though some writers may consider strategic voting acceptable, even desirable, any system in which it can be applied by individuals is not strategyproof in terms of the Gibbard-Satterthwaite impossibility theorem. However, when the strategy is applied universally by the social choice system itself and not left up to individual choosers, they are incentivized to vote sincerely utilitarian style.

Binmore[13] also assumes that, even for a welfare economy or economic democracy, voting methods are used, and hence each individual voter or chooser is allocated the power of one vote or choice thus equalizing all interpersonal comparisons.

Hillinger[14] has also made the case for utilitarian voting:

"There is, however, another branch of collective choice theory, namely utilitarian collective choice, that, instead of fiddling with Arrow's axioms, challenges the very framework within which those axioms are expressed. Arrow's framework is *ordinal* in the sense that it assumes that only the information provided by individual orderings over the alternatives are relevant for the determination of a social ordering. Utilitarian collective choice assumes that individual preferences are given as *cardinal* numbers; social preference is defined as the sum of these numbers. The fact that voting procedures are cardinal suggests that cardinal rather than ordinal collective choice theory should be relevant."

The difference between Hillinger's statement and the system considered here is that social preference is *not* defined as the sum of cardinal numbers. There is a transformation from the cardinal inputs to approval style outputs which can then be converted back into cardinal numbers if desired. Hence, the system we examine is a utilitarian approval hybrid.

Hillinger[15] advocates Evaluative Voting (EV) in which the voter assigns a value to each candidate. For example, EV-3 assigns one of the values (-1,0,+1), and then the values are summed over all candidates to determine the winner. Lorinc Mucsi[16] also supports Hillinger in his advocacy for EV-3 which allows the voter to vote for, against or remain neutral regarding each candidate. The problem with approval voting, which Hillinger claims to ameliorate, is what to do with the candidates that are neither strongly approved of or strongly disapproved of [i.e.] those in the middle. Hillinger assigns these candidates a value of zero. He[17] asserts:

"Another criticism of *AV* [Approval Voting], is due to Lawrence Ford, chair of the mathematics
department, Idaho State University, ... :

One big flaw [of *AV*] is that most voters are fairly positive of their favorites and fairly positive of those they hate, but wishy-washy in the middle. If they choose randomly for or against approval in that middle range, the whole election can become random.

Directed against *AV*, this criticism has some validity because under *AV*, not to approve a candidate is equivalent to being against him. This puts the voter in a bind of having to be for or against, when in fact he lacks the relevant information for [such] a judgment."

The use of an optimal threshold to determine which candidates get an approval style vote of +1 and which get an approval style vote of –1 clears up one of the criticisms of approval voting regarding what to do about candidates that a voter is wishy washy about. All those above threshold get a +1 vote; all those below get a –1 vote. The only ones who would get a 0 vote would be those that fell directly on the threshold.

Lehtinen[18] concludes that Arrow's Impossibility Theorem is not relevant in the final analysis: "Arrow's impossibility result and the closely related theorems given by Gibbard and Satterthwaite are unassailable as deductive proofs. However, we should not be concerned about these results because their most crucial conditions are not justifiable. Fortunately, we know that strategy-proofness is usually violated under all voting rules and that IIA does not preclude strategic voting." Contrary to Lehtinen's assertion, strategyproofness is not violated if the system itself applies the strategy instead of the individual choosers.

Optimal Threshold Social Choice

Details of the Optimal Threshold Social Choice System (OTSC) can be found in my paper, “Optimal Threshold for Selection of Candidates in Multi-Winner Elections.”[19] It can be modeled as follows:

To state the problem formally, let S be the set of all alternatives (political candidates or work-commodity bundles or a distribution of commodities and labor requirements etc.). Let W be the set chosen by society based on individual inputs. $W \subset S$. $|W| < |S|$. Inputs from the choosers are in the form of utilities. A threshold is placed in each individual chooser's utility ratings with utilities above threshold being converted to maximum approval style choices and utilities below threshold being converted to minimum approval style choices. The information processing system generates the set W based on the previously decided size of W. Summed over the choosers, the top |W| approval style choices would comprise that set which we call the “winning set”. We devise a rational method for determining which alternatives should be given approval style choices for each individual chooser based on their individual sincere preference ratings. We assume that each individual specifies a utilitarian style input composed of real numbers in the range from –1 to +1. Later we will show that any utility scale will yield the same results and that ordinal inputs can be acceptable as well after they are transformed into utilitarian style inputs.

The OTSC system processes each individual input in such a way as to find an optimal threshold T, a real number, $(-1 < T < 1)$ above which alternative ratings are converted to approval style +1 choices. Alternatives with ratings less than the optimal threshold are converted to -1 approval style choices. The optimal threshold is placed in each individual's utilitarian style input so as to maximize the expected utility of the winning set for each individual chooser.

To state the parameters formally for each individual:

Let C be the set of all candidates, c_i be a particular candidate with associated utility, u_i, U be the set of utilities corresponding to all candidates, U_a be the set of utilities above threshold and U_b be the set of utilities below threshold. Let C_a be the set of candidates above threshold and C_b be the set of candidates below threshold. Let u_a be the sum of utilities above threshold and u_b be the sum of utilities below threshold. Let n_a be the number of candidates above threshold and n_b be the number of candidates below threshold so that $n = n_a + n_b$ = total number of candidates with associated utilities.

Let V_a be a random variable which represents the utility of the winning set for each individual chooser. Then the OTSC system maximizes

$$\mathbf{E(V_a) = \sum_{i=1}^{n_a} p_i u_i}$$

In general, we define p_i as follows:

$$p_i = P[c_i \in W \,\big|\, |W \cap C_a| \geq 1] P[|W \cap C_a| \geq 1]$$

This can be interpreted as the probability that candidate i is in the winning set given that one or more above threshold candidates are in the winning set times the probability that one or more above threshold candidates are in the winning set. For the present we assume that polling information is unknown. The method easily extends to the case in which polling information is available.

The probability of the i^{th} candidate being in the winning set given that one or more above threshold candidates are in the winning set is $1/n_a$. The probability of one or more above threshold candidates being in the winning set can be expressed by the hypergeometric function. The hypergeometric function can be modeled as a ball and urn problem containing white and black balls. The candidates above threshold are identified with white balls and the candidates below threshold are identified with black balls. We posit a “picker” that picks balls one at a time out of the urn without replacement and places the balls in or out of the winning set.

The mathematical details associated with the hypergeometric function can be found in my paper, “Optimal Threshold for Selection of Candidates in Multi-Winner Elections.”

In general we have

$$\mathbf{p_i = 1 - [1- (n_a/n)][1- n_a/(n-1)]...[1- n_a/(n-i)]...[1- n_a/(n-m+1)]}$$

where $m = |W|$ and $m < n - n_a - 1$.

Therefore, the expected value of the utility associated with above threshold candidates for a particular individual voter is the following:

$$\mathbf{E(V_a) = p_i\left(\frac{1}{n_a}\right)\sum_{i=1}^{n_a} u_i}$$

and

$$\mathbf{E(V_a) = p_i(u_a/n_a).}$$

The OTSC filter does the computations for every possible threshold to determine which threshold is best i.e. which threshold results in the maximum value of expected utility of the winning set for the individual chooser under consideration. Supercomputers should have no problem with the amount of computations involved, and there are algorithms for zeroing in on the correct value for the threshold. All candidates above threshold will have their choices increased to +1, and those below threshold will be decreased to –1. Candidates whose utilities fall exactly on or close to the threshold will be set to zero. The results for all candidates will then be tallied over all choosers. In addition to the individual choice thresholds there is a social choice threshold in the voting results corresponding to the size of the winning set. All those with social choice totals above this threshold will be declared members of the winning set. **Maximizing individual voter satisfaction or utility has to do with the correct placement of the individual choice threshold for each chooser.**

The theory advanced here results in approval style choosing in the sense that individual cardinal or ordinal inputs are converted to approval style choices. Historically, approval voting is geared to selecting one candidate from a single member district. In that case it has been shown that votes should be cast for all candidates who are above average with respect to a voter's cardinal rating scale. Smith[20], proves the following: “Mean-based thresholding is optimal range-voting strategy in the limit of a large number of other voters, each random independent full-range.” Range voting is similar to utilitarian voting. Lehtinen[21] has used expected utility maximising voting behavior to indicate which candidates should be given an approval style vote in single member districts. He agrees with Smith that an approval style vote of +1 should be given to all candidates for whom their utility exceeds the average for all candidates. All others would get an AV vote of zero. For single member districts then the optimal threshold is placed at the mean of the sincere ratings for each individual. Without loss of generality we use a minimum AV style vote of -1 instead of zero.

As the threshold is raised, p_i gets smaller while u_a/n_a gets larger.We want to determine where to place the threshold so as to maximize the expected utility of those candidates above threshold for the individual voter under consideration. To simplify the discussion, let us assume, as an example, that the values of the possible utilities are uniformly spread from –1 to +1 in accordance with the spacing,

$$\frac{2}{(n - 1)}$$

and that there is one candidate corresponding to each utility. The results are easily extended to a more generalized solution since they only depend on the sum of utilities above threshold, the number of candidates above threshold, the total number of candidates and the size of the winning set.

The analysis for the OTSC system agrees with Smith and Lehtinen when the winning set contains only one winner. Since strategy is used by the OTSC system in order to maximize the expected utility for each individual, individual choosers are disincentivized from choosing insincerely or strategically. Since the system does this for them, there is no need or incentive for the individual chooser to apply strategy. Smith has shown that for a one winner outcome all ratings greater than the individual's average rating are changed to the maximum rating, and all ratings less than the average are changed to the minimum rating. Since we use maximum and minimum ratings of +1 and -1, respectively, in our analysis, this is equivalent to placing a threshold at the mean of the preference ratings and adjusting the ratings of every sincere preference above the threshold to +1 and every rating below the threshold to –1. Preference ratings falling right on the threshold can be given a 0 choice similar to Hillinger's preferred EV-3 voting method. Finally, the approval style choices for each candidate are summed over all choosers, and the candidate with the most approval style choices is declared the winner. In our initial analysis we only consider the case in which statistics regarding the choices of other choosers are unknown. It would be straighforward to take polling information into account, but that is beyond the scope of this paper. Lehtinen considers a case in which the statistics regarding other choosers are considered.

Let's do an example with m = 1 which should check with the previous results from Smith and Lehtinen.

$$p_i = 1 - \frac{n - n_a}{n}$$

Expected value of utility = $E(V_a) = p_i(u_a/n_a) = (n_a/n)(u_a/n_a) = u_a/n$

The result agrees with Smith and Lehtinen as shown in Appendix 1. In Appendix 2 we compare the results for m=1 and m=2. This graph shows that, as the winning set increases in size, everything else remaining the same, the individual chooser is more likely to achieve more utility from the winning set since more of their highly preferred alternatives are likely to be in it. Therefore, the optimal threshold can be increased.

Appendix 3 shows the results for higher values of m.

Optimal Threshold Social Choice Meets Arrow's Five Conditions

Arrow's five rational and normative conditions are

1) Unrestricted domain.
2) Positive Association of Individual and Social Values
3) Independence of Irrelevant Alternatives (IIA)
4) Citizens' Sovereignty
5) Non-dictatorship

Since all possible individual choices are under consideration, number (1) is satisfied. Number (2) is satisfied because raising an alternative in some individual's utilitarian style input from just under to just above threshold will result in that alternative's receiving one more approval style individual choice which could raise the social choice by one for that alternative putting that alternative in the winning set. Number (4) is satisfied since the OTSC system treats all alternatives and citizens in an equal and neutral manner, and number (5) is satisfied since the winning set is based only on individual inputs in such a way that no individual has any more say over the outcome than any other individual. Assume that a dictator decides that aPb. However, if the sum total of votes for b is gretare than the sum total for a, the OTSC system will decide bPa contrary to the assumption.

As for number (3), IIA, first of all utilitarian style ratings for each candidate are assumed to remain the same regardless of the composition of the alternative set. Consider Arrow's example in which one of the candidates dies and how this affects the election results using OTSC. Arrow states[22]: “Suppose that an election is held, with a certain number of candidates in the field, each individual filing his list of preferences, and then one of the candidates dies. Surely the social choice should be made by taking each of the individual's preference lists, blotting out completely the dead candidate's name, and considering only the orderings of the remaining candidates in going through the procedure of determining a winner.” Arrow implies that the voting has already occurred, but the final determination of the winner(s) has not been made. If this were the case, the OTSC Information Processing System could literally recompute all the individual thresholds and recompute the winning set thus satisfying Arrow's condition. However, there is no need to do this since the dead candidate can just be blotted out of the previously computed social results. In fact the optimal threshold will not change even if a candidate dies after the election occurs. OTSC will produce identical results for all the other candidates if the death occurs after the election takes place as is proven in Appendix 4. Iterating this process shows that the OTSC system is based on pairwise comparisons and is totally compliant with IIA. As Arrow states[23]: “Knowing the social choices made in pairwise comparisons in turn determines the entire social ordering...."

Optimal Threshold Social Choice is Strategyproof

This section refers to my paper[24], "Arrow's and Gibbard-Satterthwaite's Impossibility Theorems Revisited"

Since the data is processed in an optimal manner for each individual voter by the system itself, the voters have no incentive to misrepresent their preferences or to choose insincerely. They would either choose sincerely or the OTSC filter would process their input in such a way as to give them a suboptimal result. A social welfare function (Arrow's term) or a voting procedure (Satterthwaite's term) in which the strategy is inherent in the choosing procedure itself and applies to all choosers leads to a system in which there is no advantage to individuals to misrepresent their preference orderings or ratings. Clearly, Gibbard-Satterthwaite's theorems do not apply. The voters do not have an incentive to vote insincerely and the voting system has not led to a dictator. The strategy has been placed in the processing of the choices rather than in each individual chooser's hands. The choosers themselves are disincentivized from choosing insincerely.

The optimum strategy is to set a threshold in each individual's utilitarian style input which gives every alternative above threshold the maximum "vote" and every alternative below threshold the minimum "vote" in such a way as to maximize the expected value of utility of the social choice for each individual. This effectively turns the utilitarian style inputs into approval style "votes," but the connection with the utilitarian basis of the system is maintained

The Issue of Interpersonal Comparisons is Moot

This section refers to my paper, "Interpersonal Comparisons and Utilitarian Social Choice"[25]

Arrow[26] dwells on the fact that individual utility scales are not compatible. He compares them with the measurement of temperature which is based on arbitrary units and the arbitrary terminal points of freezing and boiling for the Celsius scale and completely different end points for the Fahrenheit scale. "Even if, for some reason, we should admit the measurability of utility for an individual, there still remains the question of aggregating the individual utilities. At best, it is contended that, for an individual, his utility function is uniquely determined up to a linear transformation; we must still choose one out of the infinite family of indicators to represent the individual, and the values of the aggregate (say a sum) are dependent on how the choice is made for each individual. In general, there seems to be no method intrinsic to utility measurement which will make the choice compatible."

Bonner[27] has discussed cardinal utility as follows: “Cardinal measurement is of little use in adding up social welfare if interpersonal comparisons cannot be made. … The scale and origin of every personal index might be different, and – what is more important – any attempt to convert them to a common basis would be open to criticism.” We show that even though the scale and origin of every personal index may be different, the OTSC method can process them in such a way that each individual's input will yield maximal results for them. Regardless of an affine linear transformation of each utility scale, the results for OTSC will be the same so that the individual choosers are free to choose any scale they want. Since this is true, any convenient scale such as the real line between –1 and +1 can be chosen without any loss of generality or arbitrariness. Or a range of choices between the integers 0 and 99 can be chosen as in range or score voting[28].

Let's unpack Bonner's statement. First, let's admit the measurability of utility for each individual. Let's say that, in general, utility can be measured as points on the real line where $-\infty < x < +\infty$ and x is a point of the real line. It's up to the individual where to place the points, including the end points, corresponding to the utilities of each candidate in the candidate set consisting of m alternatives, $\{c_1, c_2, \ldots, c_m\}$. It is proven in Appendix 5 that, for the OTSC system in particular, the results will be the same no matter which utility scale each individual chooses. There is no need to[29] “choose one out of the infinite family of indicators to represent the individual.” Consequently, Arrow's statement that “the values of the aggregate are dependent on how the choice is made for each individual” is not necessarily true. However, since any scale chosen by any individual will yield the same results, with out loss of generality we can standardize the choosing process by choosing the real line between –1 and +1.

The OTSC procedure converts an individually specified set of utilities regardless of scale to a set of approval style decisions (ones and minus ones). The ones represent the choices for alternatives in the alternative set; the minus ones represent the choices against alternatives in the alternative set. This conversion is done in such a way as to maximize the power of each individual choice. Therefore, the choice made for each individual is “compatible” since it's made using the same rationale. No matter which scale an individual chooses, they have no incentive to misrepresent their true utilities.

Amartya Sen stated in his Nobel lecture[30] “... economists came to be persuaded by arguments presented by Lionel Robbins and others (deeply influenced by "logical positivist" philosophy) that interpersonal comparisons of utility had no scientific basis. 'Every mind is inscrutable to every other mind and no common denominator of feelings is possible.' Thus, the epistemic foundations of utilitarian welfare economics were seen as incurably defective." OTSC has shown that there is a sound epistemic basis for a utility based social choice system, and, therefore, the OTSC system is in fact logical positivist.

Preference Rankings Can Be Converted to Ratings and Processed by OTSC

Arrow's preference rankings can be converted to utility scales for each individual which are then passed through the same OTSC procedure. Since the only information for rankings is of the form aPbPcPd... (or aRbRcRd...) which is interpreted as a is preferred to b, b is preferred to c etc., we can choose any utility scale as long as the preference rankings are equally spaced along that scale since that is the only information we have. We know that the choice of which scale to use is irrelevant. Let's say we choose the real line between 0 and 100. We let the top ranked candidate be placed at 100 and the lowest ranked candidate be placed at 0. The other candidates are equally spaced on the scale. Then, since an optimal threshold exists, the OTSC information processing system outputs approval style positive choices for those candidates represented by utilities above threshold and negative choices for those candidates represented by utilities below threshold for each individual. As we have shown, any affine linear transformation will not change the results of the OTSC processing system. The outputs are in the form of integers and represent the votes or choices for each alternative for each individual. Thus the output information is ordinal. The average utility of the winning set can be computed for each individual since their input utilities are known and for society as a whole since the output ordinal social rankings can also be converted back into cardinal form by averaging over the whole individual choosing population. Thus the social choice inputs and outputs can be exactly in the form Arrow assumed or in the form of individual and social utilities.

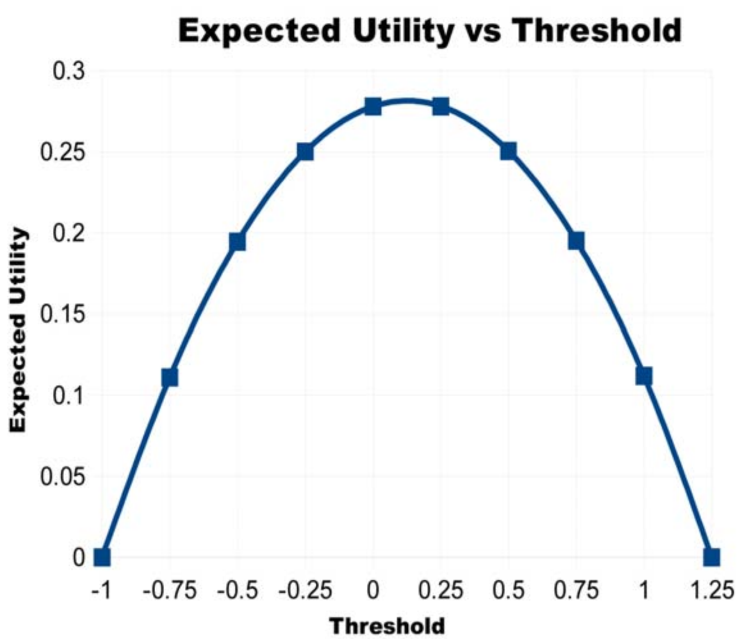
Conclusions

It has been shown that social choice is possible thus disproving both Arrow's and Gibbard-Satterthwaite's impossibility theorems which are in essence mathematical tautoligies. This has been demonstarted by evaluating the Optimal Threshold Social Choice (OTSC) system. The OTSC system accepts individual utilitarian style preference ratings as inputs and outputs approval style social choice preference rankings. The OTSC system processes the inputs in such a way as to maximize the expected utility of the social choice for each individual chooser. This is done by setting a threshold and converting the input data into approval style outputs which are then summed over all choosers thus producing social choice rankings of the alternatives. Since the OTSC converts the utilitarian style inputs to approval style outputs, OTSC is a utilitarian approval hybrid system. It has been shown that the individual choosers are disincentivized from voting insincerely since the OTSC system itself applies the optimal strategy for each individual input. Any use of strategy by individual choosers would result in a suboptimal outcome for them. Also the issue of interpersonal camparisons is moot because any affine linear transformation of an individual's utility scale will produce the same results. Finally, if inputs are specified as preference rankings rather than ratings, the rankings can be converted to utility style ratings which can then be processed by the OTSC system. The outputs which are in the form of rankings can also be converted back to ratings since utility information for each individual chooser is known. Based on the social choice, utilities can be computed for each individual or averaged for society as a whole.

Arrow's and Gibbard-Satterthwaite's impossibility theorems have been thought to negate the possibility of social democracy either in terms of welfare economics or direct democracy leaving only capitalist economics and representative democracy with a sound epistemic basis. The work presented here proves that utilitarian based social choice in fact does have a sound scientific basis.

Appendix 1

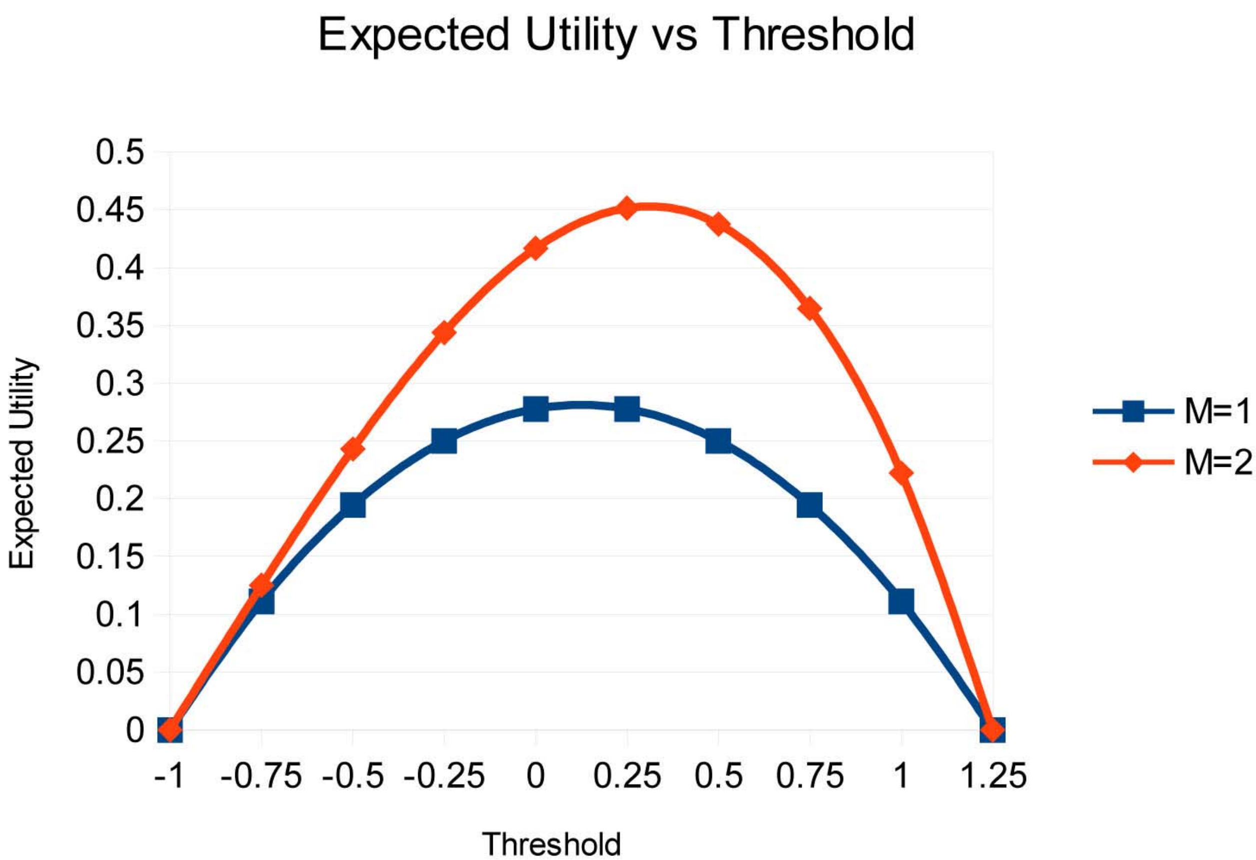
When there is one member in the winning set, expected social utility for an individual chooser is a maximum when the threshold is close to
$u_i = 0$, $n_a = (n-1)/2$. This agrees with the former analysis by Smith and Lehtinen since the threshold is placed at the mean. The maximum value of expected utility can be made to occur arbitrarily close to a threshold of zero by increasing n. The graph is as follows:



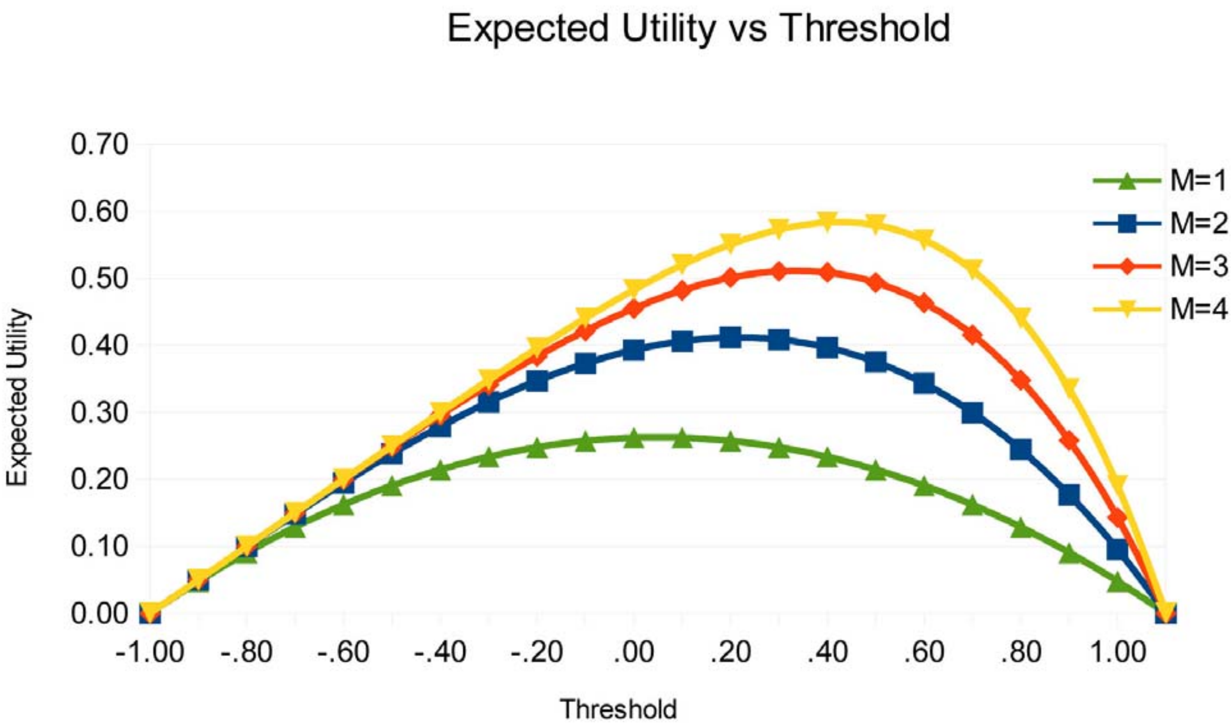
<u>Appendix 2</u>

We can see that the peak has shifted to the right and upwards indicating that the threshold for which expected average utility is maximum has shifted up towards greater utilities and the expected average utility at that threshold is greater.

As m increases, the individual chooser should derive increased utility or satisfaction from the winning set since one or more of their above threshold candidates are more likely to become part of the winning set W.



Appendix 3



Appendix 4

Theorem: For OTSC, if a candidate drops out of an election after voting has occurred, the results of the election will not be changed

Proof:

For some particular voter let the expected value of above threshold average utility be

$$\mathbf{E}(\mathbf{V}_a) = \mathbf{p}\left(\frac{\mathbf{1}}{\mathbf{n}_a}\right)\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}_a}\mathbf{u}_i$$

Assume candidate j, an above threshold candidate, drops out after votes are cast.

Then, expected value of above threshold average utility is

$$\left(\frac{\mathbf{p}}{\mathbf{n}_a - \mathbf{1}}\right)(\mathbf{u}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_{j-1} + \mathbf{u}_{j+1} + \ldots + \mathbf{u}_{n_a})$$

Let u_1 be the least above threshold value. Then the social choice may be changed if the optimal threshold is raised to exclude this value.

That would be true if

$$\frac{p\left(\sum_{i=2,i\neq j}^{n_a} u_i\right)}{n_a-2} > \frac{p\left(\sum_{i=1,i\neq j}^{n_a} u_i\right)}{n_a-1}$$

But

$$\left(\frac{n_a-1}{n_a-2}\right)\left[\sum_{i=2,i\neq j}^{n_a} p_i u_i\right] \ngtr \left[\sum_{i=1,i\neq j}^{n_a} p_i u_i\right]$$

By inspection.

Therefore, the above threshold value of average expected utility is not increased by raising the threshold.

Assume the threshold is lowered if an above threshold candidate drops out.

We renumber the expected utilities such that u_1 is the value just below threshold, so that there are n_a+1 values above threshold.

Assume:

$$\frac{p\sum_{i=1,i\neq j}^{n_a+1} u_i}{n_a} > \frac{p\sum_{i=2,i\neq j}^{n_a+1} u_i}{n_a-1}$$

$$\left(\frac{n_a-1}{n_a}\right)\left(\sum_{i=1,i\neq j}^{n_a+1} u_i\right) > \sum_{i=2,i\neq j}^{n_a+1} u_i$$

$$\left(\frac{n_a-1}{n_a}\right)\left(\sum_{i=1,i\neq j}^{n_a+1} u_i\right) > \left(\frac{n_a-1}{n_a}+\frac{1}{n_a}\right)\sum_{i=2,i\neq j}^{n_a+1} u_i$$

subtract

$$\left(\frac{\mathbf{n_a-1}}{\mathbf{n_a}}\right)\sum_{i=2,i\neq j}^{n_a+1}\mathbf{u_i}$$

from both sides

$$\left(\frac{\mathbf{n_a-1}}{\mathbf{n_a}}\right)\mathbf{u_1} > \left(\frac{\mathbf{1}}{\mathbf{n_a}}\right)\sum_{i=2,i\neq j}^{n_a+1}\mathbf{u_i}$$

$$(\mathbf{n_a-1})\mathbf{u_1} > \sum_{i=2,i\neq j}^{n_a+1}\mathbf{u_i}$$

But this is a contradiction since $u_1 < u_i$ for $2 \leq i \leq n+1$ and there are (n-1) terms on each side.

Therefore, the above threshold value of average expected utility is not increased by lowering the threshold.

Assume candidate j, a below threshold candidate, drops out after votes are cast. The threshold would not change by definition.

Since the optimal threshold doesn't have to be recomputed if a candidate drops out after the votes are cast by the voters, Arrow's IIA condition is preserved and the vote tally remains the same as if the candidate had just been blotted out of the election results. If the candidate who dropped out was in the winning set, the candidate with the highest vote total who was not in the winning set would then be elevated to it.

Appendix 5

To prove: Given any arbitrary individual utility scale consisting of preference ratings as inputs, the social choice results, when processed by the OTSC, will be the same as they would be for any affine linear transformation of that scale.

For some particular voter let the expected value of above threshold average utility be

$(p_2u_2 +\ldots + p_nu_n)/(n-1)$

where $p_ju_j < p_{j+1}u_{j+1}$ for $2 \leq j \leq n-1$ and the optimal threshold is just under p_2u_2. We perform a linear affine transformation of the form f(x) = ax +b (a nd b integers) and assume that the optimal threshold will move down from just under p_2u_2 to just under p_1u_1 so that the above threshold average utility is now

$(p_1u_1 + \ldots + p_nu_n)/(n-1)$

We assume:

$$\frac{\sum_1^n (au_j + b)}{n} > \frac{\sum_2^n (au_j + b)}{n-1}$$

We know:

$$\frac{\sum_1^n (au_j + b)}{n} = \frac{\sum_1^n (au_j) + nb}{n} = \frac{a\sum_1^n (u_j)}{n} + b$$

$$\frac{\sum_2^n (au_j + b)}{n-1} = \frac{\sum_2^n (au_j) + (n-1)b}{n-1} = \frac{a\sum_2^n (u_j)}{n-1} + b$$

So is

$$\frac{a\sum_1^n (u_j)}{n} + b > \frac{a\sum_2^n (u_j)}{n-1} + b$$

Subtracting b from both sides and dividing by a we have

$$\frac{\sum_1^n (u_j)}{n} > \frac{\sum_2^n (u_j)}{n-1}$$

However, we know that

$$\frac{\sum_1^n (u_j)}{n} < \frac{\sum_2^n (u_j)}{n-1}$$

because by definition the optimal threshold is placed just under the utility such that the average utility above threshold is a maximum.
So the assumption is not true.

Similarly, if the average above threshold utility is $(p_1u_1 + \ldots + p_nu_n)/n$, we show that applying an affine linear transformation and assuming that the optimal threshold moves up to just under p_2u_2 is false.

Assume that

$$\frac{\sum_2^n (au_j + b)}{n-1} > \frac{\sum_1^n (au_j + b)}{n}$$

Then

$$\frac{\sum_2^n (au_j + b)}{n-1} = \frac{\sum_2^n (au_j) + (n-1)b}{n-1} = \frac{a\sum_2^n (u_j)}{n-1} + b$$

and

$$\frac{\sum_1^n (au_j + b)}{n} = \frac{\sum_1^n (au_j) + nb}{n} = \frac{a\sum_1^n (u_j)}{n} + b$$

Therefore,

$$\frac{a\sum_2^n (u_j)}{n-1} + b > \frac{a\sum_1^n (u_j)}{n} + b$$

and

$$\frac{a\sum_2^n (u_j)}{n-1} > \frac{a\sum_1^n (u_j)}{n}$$

$$\frac{\sum_2^n (u_j)}{n-1} > \frac{\sum_1^n (u_j)}{n}$$

But we know that,

$$\frac{\sum_1^n (u_j)}{n} > \frac{\sum_2^n (u_j)}{n-1}$$

by definition of the optimal threshold and the assumption is false. QED.

Therefore, an affine linear transformation does not change the placement of the optimal threshold.

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